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Schwarzian Derivatives and Differential Equations

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0 Introduction

In this note, we write down explicitly the orbits of simple differential equations by groups of diffeomorphisms or contact diffeomorphisms using multi-dimensional Schwarzian derivatives or contact Schwarzian derivatives. We give a definition of contact Schwarzian derivatives which is equivalent to that of Fox [2]. For the lowest dimensional case, the definition is given in [7], [5].

1 System of second-order PDE's

In the section, we consider the simple system of second-order partial differential equations with m independent variable and n dependent variables

$$\frac{\partial^2 y_k}{\partial x_i \partial x_j} = 0 \quad \text{for all } 1 \leq i \leq j \leq m, 1 \leq k \leq n. \quad (1)$$

The number of equations in (1) is equal to $mn(m+1)/2$. Put

$$\mathbf{y} = (y_1, y_2, \dots, y_n).$$

Then the system (1) is written simply as

$$\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j} = 0 \quad \text{for all } 1 \leq i \leq j \leq m.$$

Put

$$\mathbf{x} = (x_1, x_2, \dots, x_m), \quad p_{ij} = \frac{\partial y_i}{\partial x_j} \quad (1 \leq i \leq n, 1 \leq j \leq m),$$

and let $\mathbf{p} = (p_{ij})$ be the $n \times m$ matrix whose (ij) -component is equal to p_{ij} .

Let K be \mathbb{R} or \mathbb{C} and let $\phi : K^{m+n} \rightarrow K^{m+n}$ be a nondegenerate map (diffeomorphism) given by

$$\phi : \mathbf{z} = (z_1, \dots, z_{m+n}) \rightarrow (Z_1, \dots, Z_{m+n}).$$

Put $\ell = m + n$. According to Yoshida[13], [14] or Sasaki [6], (multi dimensional) Schwarzian derivative $S_{ij}^k(\phi)$ for $1 \leq i, j, k \leq \ell$ is given by

$$S_{ij}^k(\phi) = \sum_{p=1}^{\ell} \frac{\partial^2 Z^p}{\partial z^i \partial z^j} \frac{\partial z^k}{\partial Z^p} - \frac{1}{\ell+1} \sum_{p,q=1}^{\ell} \left(\delta_i^k \frac{\partial^2 Z^p}{\partial z^q \partial z^j} \frac{\partial z^q}{\partial Z^p} + \delta_j^k \frac{\partial^2 Z^p}{\partial z^q \partial z^i} \frac{\partial z^q}{\partial Z^p} \right),$$

(cf. also Gunning[3], Kobayashi-Ochiai[4]). Clearly $S_{ij}^k(\phi) = S_{ji}^k(\phi)$ and further they satisfy the canonical-form relation

$$\sum_{k=1}^{\ell} S_{ik}^k(\phi) = 0 \quad \text{for } i = 1, 2, \dots, \ell.$$

So the number of Schwarzian derivatives is $(\ell - 1)\ell(\ell + 2)/2$.

If we regard

$$(z_1, z_2, \dots, z_{m+n}) = (x_1, \dots, x_m, y_1, \dots, y_n),$$

the diffeomorphism ϕ maps the system (1) to another system of partial differential equations

$$\frac{\partial^2 y_k}{\partial x_i \partial x_j} = f^{\phi k}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p}) \quad \text{for all } 1 \leq i \leq j \leq m, 1 \leq k \leq n. \quad (2)$$

where $f^{\phi^k}_{ij} = f^{\phi^k}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p})$ are smooth functions of $m + n + mn$ variables. The set of the system of equations $\frac{\partial^2 y_k}{\partial x_i \partial x_j} = f^{\phi^k}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p})$ for all diffeomorphisms ϕ is the orbit of the system of equations $\frac{\partial^2 \mathbf{y}}{\partial x_i \partial x_j} = 0$ by the diffeomorphism group $\text{Diff}(\mathbb{R}^{m+n})$. This is an answer to the problem of the paper [8].

In the following Theorem 1, we write down explicitly the functions $f^{\phi^k}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p})$ by using the Schwarzian derivatives $S^k_{ij}(\phi)$.

The case for $m = n = 1$ is explained in [7] and higher dimensional cases are left unsolved.

Theorem 1 *By the inverse diffeomorphism ϕ^{-1} , the system of 2O-PDE*

$$\left\{ \frac{\partial^2 Y_k}{\partial X_i \partial X_j} = 0 \quad (1 \leq i \leq j \leq m, 1 \leq k \leq n) \right\}$$

is mapped to a system of 2O-PDE

$$\left\{ \frac{\partial^2 y_k}{\partial x_i \partial x_j} = f^{\phi^k}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p}) \quad (1 \leq i \leq j \leq m, 1 \leq k \leq n) \right\}$$

where

$$\begin{aligned} f^{\phi^k}_{ij}(\mathbf{x}, \mathbf{y}, \mathbf{p}) = & -S^{m+k}_{ij} + \sum_{s=1}^m S^s_{ij} p_{ks} - \sum_{t=1}^n (S^{m+k}_{m+tj} p_{ti} + S^{m+k}_{im+t} p_{tj}) \\ & + \sum_{s=1}^m \sum_{t=1}^n (S^s_{m+tj} p_{ks} p_{ti} + S^s_{im+t} p_{ks} p_{tj}) - \sum_{s,t=1}^n S^{m+k}_{m+t m+s} p_{ti} p_{sj} \\ & + \sum_{s=1}^m \sum_{u,t=1}^n S^s_{m+u m+t} p_{ks} p_{ui} p_{tj}. \end{aligned}$$

This is an extension of the following result (cf. [7]).

Example 1 Let $\phi : K^2 \rightarrow K^2$ given by $\phi(x, y) = (X(x, y), Y(x, y))$ be a diffeomorphism. By the inverse diffeomorphism ϕ^{-1} ,

$$Y''(X) = 0$$

is mapped to

$$y''(x) = -S_{11}^2 + 3S_{11}^1 y' - 3S_{22}^2 (y')^2 + S_{22}^1 (y')^3.$$

2 System of third-order PDE's

In this section, we consider the simple system of third order partial differential equations of n independent variables and one dependent variable

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0 \quad \text{for all } 1 \leq i \leq j \leq k \leq m,$$

where $y = y(x_1, \dots, x_m)$. The space $K^{2m+1} = \{\mathbf{x}, y, \mathbf{p}\}$ has the natural contact form $\eta = dy - \sum_{i=1}^m p_i dx_i$. For $1 \leq i \leq j \leq m$, put $q_{ij} = \frac{\partial^2 y}{\partial x_i \partial x_j}$ and put $\mathbf{q} = \{q_{11}, q_{12}, \dots, q_{mm}\}$. Then \mathbf{q} consists of $\frac{1}{2}m(m+1)$ functions.

Let $\varphi : K^{2m+1} \rightarrow K^{2m+1}$ be a contact diffeomorphism with a nonvanishing function f such that $\varphi^* \eta = f \eta$. The contact diffeomorphism φ maps the equations $\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0$ to another system of partial differential equations

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = f^\varphi_{ijk}(\mathbf{x}, y, \mathbf{p}, \mathbf{q}) \quad \text{for } 1 \leq i \leq j \leq k \leq m.$$

Here f^φ_{ijk} are smooth functions of $2m+1 + \frac{1}{2}m(m+1) = \frac{m^2+5m+2}{2}$ variables.

We define (multi-dimensional) contact Schwarzian derivatives $C_{ij}^k(\varphi)$ (Definition 2.1) which is equivalent to that of Fox [2]. For $m = 1$, the definition is given in [7], [5].

In the following Theorem 2, we write down explicitly the function $f_{ij}^{\varphi}(\mathbf{x}, y, \mathbf{p}, \mathbf{q})$ by using the contact Schwarzian derivatives $C_{ij}^k(\varphi)$. This extends a result of [7] for the case of $m = 1$.

Contact Schwarzian Derivatives

On the contact space $K^{2n+1} = \{(\mathbf{x}, y, \mathbf{p})\}$, the total differential $\frac{d}{dx_k}$ is defined by

$$\frac{d}{dx_k} = \frac{\partial}{\partial x_k} + p_k \frac{\partial}{\partial y}.$$

For a function A , the total differential of A is expressed as

$$A_{;k} = \frac{dA}{dx_k} = A_{x_k} + A_y p_k.$$

Let $\varphi : K^{2n+1} \rightarrow K^{2n+1}$ be a contact transformation given by

$$\varphi(x_1, \dots, x_n, y, p_1, \dots, p_n) = (X_1, \dots, X_n, Y, P_1, \dots, P_n).$$

Corresponding to φ , define $(n \times n)$ matrices $X_{;}$, X_p , $P_{;}$, P_p by

$$(X_{;})_{ij} = X_{i;j}, \quad (X_p)_{ij} = \frac{\partial X_i}{\partial p_j}, \quad (P_{;})_{ij} = P_{i;j}, \quad (P_p)_{ij} = \frac{\partial P_i}{\partial p_j},$$

and define $(2n) \times (2n)$ -matrix \mathcal{X}_{φ} by

$$\mathcal{X}_{\varphi} = \begin{pmatrix} X_{;} & X_p \\ P_{;} & P_p \end{pmatrix}.$$

Then $(\mathcal{X}_{\varphi})^{-1} = \mathcal{X}_{\varphi^{-1}} = \begin{pmatrix} x_{;} & x_p \\ p_{;} & p_p \end{pmatrix}$. For $1 \leq r, s \leq 2n$, let α_{rs} be the (r, s) -component of the matrix \mathcal{X}_{φ} ; $\alpha_{rs} = (\mathcal{X}_{\varphi})_{rs}$. Let β_{rs} be the (r, s) -component of the matrix $\mathcal{X}_{\varphi^{-1}}$; $\beta_{rs} = (\mathcal{X}_{\varphi^{-1}})_{rs}$. Then β_{st} is equal to α_{st} with replacing small letters x, p with capital letters X, P .

For $1 \leq r, s, t \leq 2n$, put

$$\alpha_{rst} = \begin{cases} \alpha_{rs;t} & \text{if } 1 \leq t \leq n \\ \frac{\partial \alpha_{st}}{\partial p_{t-n}} & \text{if } n \leq t \leq 2n \end{cases}, \quad \mu_{rs}^t = \sum_{u=1}^{2n} \beta_{tu} \alpha_{urs}$$

Then, for $1 \leq i, j, k \leq 2n$, the contact Schwarzian derivatives $C_{ij}^k = C_{ij}^k(\varphi)$ is defined by

$$C_{ij}^k = \frac{1}{2} (\mu_{ij}^k + \mu_{ji}^k) - \frac{1}{2(2n+1)} \sum_{s=1}^{2n} (\delta_{ik} \mu_{js}^s + \delta_{ik} \mu_{sj}^s + \delta_{jk} \mu_{is}^s + \delta_{jk} \mu_{si}^s).$$

Now we have the following theorem;

Theorem 2 By the inverse contact diffeomorphism φ^{-1} , the system of 3O-PDE

$$\left\{ \frac{\partial^3 Y}{\partial X_i \partial X_j \partial X_k} = 0 \quad (1 \leq i \leq j \leq k \leq m) \right\}$$

is mapped to a system of 3O-PDE

$$\left\{ \frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = f_{ijk}^\varphi(\mathbf{x}, y, \mathbf{p}, \mathbf{q}) \quad (1 \leq i \leq j \leq k \leq m) \right\}$$

where

$$\begin{aligned} & f_{ijk}^\varphi(\mathbf{x}, y, \mathbf{p}, \mathbf{q}) \\ &= -C_{jk}^{m+i} + \sum_{\ell=1}^m C_{jk}^\ell q_{i\ell} - \sum_{\ell=1}^m C_{m+\ell k}^{m+i} q_{j\ell} - \sum_{\ell=1}^m C_{jm+\ell}^{m+i} q_{\ell k} \\ &+ \sum_{\ell,r=1}^m C_{jm+r}^\ell q_{i\ell} q_{rk} + \sum_{\ell,r=1}^m C_{m+r k}^\ell q_{i\ell} q_{rj} - \sum_{\ell,r=1}^m C_{m+\ell m+r}^{m+i} q_{\ell j} q_{rk} + \sum_{\ell,r,s=1}^m C_{m+\ell m+r}^s q_{is} q_{\ell j} q_{rk}. \end{aligned}$$

This is an extension of the following result ([7],[5]).

Example 2 Let $\varphi : K^3 \rightarrow K^3$ given by $\varphi(x, y, p) = (X(x, y, p), Y(x, y, p), P(x, y, p))$ be a contact diffeomorphism. By the inverse contact diffeomorphism φ^{-1} ,

$$Y'''(X) = 0$$

is mapped to

$$y'''(x) = -C_{11}^2 + 3C_{11}^1 y'' - 3C_{22}^2 (y'')^2 + C_{22}^1 (y'')^3.$$

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